String Theoretical Interpretation for
Finite N Yang-Mills Theory in Two-Dimensions

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Abstract

We discuss the equivalence between a string theory and the two-dimensional Yang-Mills theory with $SU(N)$ gauge group for finite N. We find a sector which can be interpreted as a sum of covering maps from closed string world-sheets to the target space, whose covering number is less than N. This gives an asymptotic expansion of $1/N$ whose large N limit becomes the chiral sector defined by D. Gross and W. Taylor. We also discuss that the residual part of the partition function provides the non-perturbative corrections to the perturbative expansion.

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1 Introduction

Recent studies on string theories have shown that exploring the relationship among string theories, gauge theories and matrix models is an important issue for understanding the string theories themselves. As typical examples, the duality between gauge theories and string theories [1] is realized as the AdS/CFT correspondence [2] (for a recent review, see also Ref. [3]), while the large N reduction of gauge theories realizes the gauge/matrix correspondence [4], and large N matrix models have been proposed as candidates for non-perturbative definition of the string/M theory [5, 6].

Explicit analysis is rather tractable in lower dimensions. The duality between two-dimensional non-critical string theory and the $c = 1$ matrix model is one of the most successful examples. It has recently been found that the non-perturbative effect predicted from the $c = 1$ matrix model indeed describes D-branes in the non-critical string theory [7].[1] As for the gauge/string equivalence, Gross and Taylor have attempted in their seminal works [10, 11, 12] to uncover the relationship between the two-dimensional Yang-Mills theory (YM$_2$) and a string theory, to which we pay particular attention in this paper. The statement of this equivalence is that the free energy of Yang-Mills theory is identified with a partition function of a string theory on the target space $\mathcal{M}$ with the string coupling $1/N$ and the string tension λ [10]. Strong evidence for this conjecture is given in Refs. [10]–[12] by showing that the partition function of the large N YM$_2$ can be interpreted as a sum of all maps without folds from a set of closed string world-sheets to the target space. (For review and further developments, see, e.g. Refs. [14]–[21].) This means that YM$_2$ describes at least a perturbative closed string theory in the sense that the asymptotic $1/N$ expansion of the free energy corresponds to a perturbative expansion of a closed string theory.

In this paper, we are interested in a string theoretical interpretation of the finite N YM$_2$. The crucial point is that we know the exact expression of the partition function of the YM$_2$ for finite N. According to the modern understanding of critical and/or non-critical string theories, a closed string theory contains not only closed strings themselves but also open string degrees of freedom emerging through D-branes, which are the non-perturbative objects characterized by a factor e^{-1/g_s} in free energy of the theory. It is natural to wonder whether this is the case or not for a string theory defined through YM$_2$. We show that the partition function of YM$_2$ takes a form which enable us to conclude that this is indeed the case by dividing the partition function into two sectors in a definite way. One of these sectors turn out to be interpreted as a sum of maps from closed string world-sheets, whose large N limit becomes the chiral sector [11] of the large N theory. Furthermore, we argue that the other sector has properties that is consistent with non-perturbative effects of a closed string theory.

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[1]For other discussions for the matrix/string correspondence, see, e.g. Refs. [8, 9].

This paper is organized as follows. In the next section we briefly review the relationship between YM$_2$ and a string theory. We describe here the basic ingredients and concepts needed to understand the relationship. In section 3, we examine the partition function of the finite N YM$_2$ and discuss the string theoretical interpretation from the perturbative and the non-perturbative points of view. Section 4 is devoted to summarize our results and discussions. In the appendix, we review a geometric interpretation of the large N YM$_2$ partition function.

2 YM$_2$/string correspondence

In this section, we briefly review the duality between YM$_2$ and a string theory [10]–[12]. First, we will explain the technique for translating the partition function of YM$_2$ into an expression of which we can give a geometrical interpretation. We will next show that the resulting expression can further be regarded as a perturbative expansion of a closed string theory, especially in the case of large N theory.

Let us consider the Yang-Mills theory with $SU(N)$ gauge symmetry on a two-dimensional (Euclidean) manifold $\mathcal{M}$ with genus G. This theory is exactly solvable and the partition function is evaluated as [22]

$$
\begin{aligned}
Z_{\mathrm{YM}}(G, \lambda A, N) &= \int [dA_\mu] \exp\left[-\frac{1}{4g_{YM}^2}\mathrm{tr}\int d^2x\sqrt{g}F_{\mu\nu}F^{\mu\nu}\right] \\
&= \sum_R (\dim R)^{2-2G}\exp\left[-\frac{\lambda A}{2N}C_2(R)\right],
\end{aligned}
\tag{2.1}
$$

where $\lambda \equiv g_{\mathrm{YM}}^2 N$ is the 't Hooft coupling, A is the area of the manifold $\mathcal{M}$, R is an irreducible representation of the gauge group, and $\dim R$ and $C_2(R)$ are the dimension and the quadratic Casimir of R, respectively.

In the following, we will rewrite the partition function (2.1) in the language of the symmetry group. The irreducible representation of $SU(N)$ can be classified by the Young diagrams with the rows less than N, and that of the symmetry group S_n can be classified by the Young diagrams with n boxes. Thus, we will use the same notation R to express the Young diagram corresponding to the irreducible representation of $SU(N)$ and S_n.

One might conclude that the partition function in the large N theory is given by

$$
\sum_{n=1}^{\infty}\sum_{R\in Y_n}(\dim R)^{2-2G}\exp\left[-\frac{\lambda A}{2N}C_2(R)\right],
\tag{2.2}
$$

where Y_n denotes the set of Young diagrams with n boxes. However, as discussed in Ref. [11], this is a half of the full large N theory, since the contribution to the partition function of a Young diagram R with a finite number of boxes is the same as that of the conjugate

representation $\bar{R}$. To take the both contributions into account, the authors of Ref. [11] conjecture that the $1/N$ expansion of the partition function is evaluated from

$$Z_{\mathrm{YM}}(G, \lambda A, N) = \sum_{n=1}^{\infty} \sum_{\tilde{n}=1}^{\infty} \sum_{R \in Y_n} \sum_{S \in Y_{\tilde{n}}} (\dim \overline{S}R)^{2-2G} e^{-\frac{\lambda A}{2N} C_2(\overline{S}R)}, \tag{2.3}$$

where $\bar{S}R$ is called the composite diagram made from Young diagrams R and S, whose column length is

$$\begin{cases} N - \tilde{c}_{L+1-i}, & i \leq L, \\ c_{i-L}, & i > L, \end{cases}$$

where c_i and $\tilde{c}_i$ are the length of i-th column of R and S, respectively, and L is the length of the first row of the representation S. Since the dimension and the quadratic Casimir of the composite representation are

$$\dim \overline{S}R = \dim R \dim S \left[1 + \mathcal{O}\left(1/N^2\right)\right], \tag{2.4}$$

$$C_2(\overline{S}R) = C_2(R) + C_2(S) + \frac{2n\tilde{n}}{N}, \tag{2.5}$$

we can see that the partition function of the full theory is factorized into two copies of a chiral sectors [11],

$$Z^+(G, \lambda A, N) \equiv \sum_{n=1}^{\infty} \sum_{R \in Y_n} (\dim R)^{2-2G} \exp\left[-\frac{\lambda A}{2N} C_2(R)\right], \tag{2.6}$$

except for the $\mathcal{O}(1/N^2)$ correction of the dimension and a coupling term $e^{-\frac{\lambda A n \tilde{n}}{N^2}}$ from the quadratic Casimir. In the following analysis, we concentrate the discussion only on the chiral sector. For a geometrical interpretation of the full partition function (2.3), see Ref. [12].

The dimension and the quadratic Casimir of the irreducible representation R are evaluated by using the character of the symmetric group S_n [12];

$$\dim R = \frac{1}{n!} \sum_{\sigma \in S_n} \chi_R(\sigma) N^{K_\sigma} \equiv \frac{d_R N^n}{n!} \frac{\chi_R(\Omega_n)}{d_R}, \tag{2.7}$$

$$C_2(R) = Nn + 2 \sum_{p \in T_2} \frac{\chi_R(p)}{d_R} - \frac{n^2}{N}, \tag{2.8}$$

where $\chi_R(\sigma)$ is the character of $\sigma \in S_n$ in the representation R, d_R is the dimension of the representation R of S_n, T_2 is the set of transpositions, and $\Omega_n \equiv \sum_{\sigma \in S_n} \sigma / N^{n-K_\sigma}$ where K_σ is the number of cycles in σ, which is an element of the group ring of S_n. Substituting these into the partition function (2.6), one will find that the partition function of the chiral sector

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becomes[2]

$$Z^+(G, \lambda A, N) = \sum_{n=1}^{\infty} e^{-\frac{n\lambda A}{2}} \sum_{i,t,h=0}^{\infty} \frac{(-1)^i (\lambda A)^{i+t+h}}{i!\, t!\, h!} \left(\frac{n(n-1)}{2}\right)^t \left(\frac{n}{2}\right)^h N^{n(2-2G)-i-2t-2h}$$

$$\times \sum_{s_1,t_1,\cdots,s_G,t_G \in S_n} \sum_{p_1,\cdots,p_i \in T_2} \frac{1}{n!} \delta(p_1 \cdots p_i \Omega_n^{2-2G} \prod_{j=1}^{G} s_j t_j s_j^{-1} t_j^{-1}). \qquad (2.9)$$

This is the expression of the partition function in which a geometrical interpretation is possible. In fact, (2.9) can be interpreted as a sum of covering maps from a set of two dimensional manifolds to the target space. The indices i, t and h is regarded as the number of single branch points, tubes, and contracted handles [11], respectively. Furthermore the delta function in (2.9) provides a sum of the symmetry factor $|S_\nu|$ of a given covering map ν in a set of covering maps $\Sigma(G, n, i)$;

$$\sum_{s_1,t_1,\cdots,s_G,t_G \in S_n} \sum_{p_1,\cdots,p_i \in T_2} \frac{1}{n!} \delta(p_1 \cdots p_i \Omega_n^{2-2G} \prod_{j=1}^{G} s_j t_j s_j^{-1} t_j^{-1}) = \sum_{\nu \in \Sigma(G,n,i)} 1/|S_\nu|. \qquad (2.10)$$

Details of the geometrical interpretation are provided in the appendix.

In order to obtain the free energy $W^+(G, \lambda A, N)$ which is the logarithm of the partition function, it is enough to take restricted sums in the partition function which contain a set of connected part of the covering maps $\tilde{\Sigma}(G, n, i)$;

$$W^+(G, \lambda A, N) = \sum_{n=1}^{\infty} e^{-\frac{n\lambda A}{2}} \sum_{i,t,h=0}^{\infty} \frac{(-1)^i (\lambda A)^{i+t+h}}{i!\, t!\, h!} \left(\frac{n(n-1)}{2}\right)^t \left(\frac{n}{2}\right)^h$$

$$\times N^{n(2-2G)-i-2t-2h} \sum_{\nu \in \tilde{\Sigma}(G,n,i)} 1/|S_\nu|. \qquad (2.11)$$

To complete our discussion, we must further show that the above geometrical interpretation is also "string theoretical", that is, the expression (2.11) can be interpreted as a sum of maps from closed string world-sheets to the target space with a weight corresponding to a world-sheet action. First of all, the term $e^{-\frac{n\lambda A}{2}}$ is naturally interpreted as a contribution

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[2]We have used the formulae

$$\sum_{\sigma \in T} \chi_R(\sigma) \chi_R(\rho) = d_R \sum_{\sigma \in T} \chi_R(\sigma \rho) \quad (T : \text{a conjugacy class in } S_n),$$

$$\sum_{s,t \in S_n} \frac{\chi_R(sts^{-1}t^{-1})}{d_R} = \left(\frac{n!}{d_R}\right)^2, \quad \text{and} \quad \delta(\sigma) = \frac{1}{n!} \sum_{R \in Y_n} d_R \chi_R(\sigma).$$

The definition of the delta function in S_n is

$$\delta(\sigma) = \begin{cases} 1 & \sigma = 1 \\ 0 & \text{others} \end{cases} \quad \text{for } \sigma \in S_n.$$

from a Nambu-Goto type world-sheet action if the 't Hooft coupling λ is identified with the string tension. Next, we would like to interpret $1/N$ as a string coupling g_s, however, there is a subtlety here. Since the Euler characteristic of a closed string world-sheet is even number, we must show that (2.6) contains only even powers of $1/N$. We will see below that (2.11) is indeed the case although it seems to contain odd powers of $1/N$.

As mentioned in Ref. [10], if N is large enough there exists the "transpose representation" R^T for any irreducible representation R in the sum in (2.1). Here, the Young diagram R^T is defined by exchanging the rows and columns of the original diagram R. When the dimension and the quadratic Casimir of the representation R are given as (2.7) and (2.8), respectively, those of R^T are [10]

$$\dim R^T = \frac{N^n}{n!} \sum_{\sigma \in S_n} (-N)^{K_\sigma - n} \chi_R(\sigma), \tag{2.12}$$

$$C_2(R^T) = Nn - 2 \sum_{p \in T_2} \frac{\chi_R(p)}{d_R} - \frac{n^2}{N}. \tag{2.13}$$

Thus, by rewriting the partition function (2.6) as

$$
\begin{aligned}
&Z^+(G, \lambda A, N) \\
&= \sum_{n=1}^{\infty} \sum_{R \in Y_n} \frac{1}{2} \left\{ (\dim R)^{2-2G} \exp\left[-\frac{\lambda A}{2N} C_2(R) \right] + (\dim R^T)^{2-2G} \exp\left[-\frac{\lambda A}{2N} C_2(R^T) \right] \right\},
\end{aligned}
\tag{2.14}
$$

we can see that only even powers of $1/N$ survive. For example, let us write down the partition function for $G = 1$ in the language of S_n as follows:

$$
\begin{aligned}
&Z^+(G = 1, \lambda A, N) \\
&= \sum_{n=1}^{\infty} e^{-\frac{n\lambda A}{2}} \sum_{c,t,h=0}^{\infty} \frac{(\lambda A)^{2c+t+h}}{(2c)! \, t! \, h!} \left(\frac{n(n-1)}{2} \right)^t \left(\frac{n}{2} \right)^h N^{-2(c+t+h)} \\
&\quad \times \sum_{s,t \in S_n} \sum_{p_1,\cdots,p_{2c} \in T_2} \frac{1}{n!} \delta(p_1 \cdots p_{2c} s t s^{-1} t^{-1}).
\end{aligned}
\tag{2.15}
$$

As announced, the exponents of N contain only even numbers. In (2.15), the index c can be regarded as the number of cuts on the world-sheets. It is straightforward to carry out the same calculation for an arbitrary number of G.

3 Dual string theory for the finite N YM$_2$

In the previous section, we have reviewed the geometrical and string theoretical interpretation of YM$_2$ given by Gross and Taylor. From their analysis, it seems to be natural to conclude

that YM$_2$ describes a closed string theory. However, there are two subtle points in the analysis if we consider the non-perturbative corrections; first, the way to decompose a Young diagram T into S and R as $T = \overline{S}R$ is not unique, and second, in (2.6), there are diagrams in Y_n that do not correspond to any irreducible representation of $SU(N)$ when n is larger than N. Although we do not face these problems in large N limit, we must start with finite N YM$_2$ to evaluate the $1/N$ perturbative expansion and the non-perturbative corrections of the partition function (2.1). Actually, a geometrical interpretation of the finite N YM$_2$ is also possible [13]. However, as we will mention later, this interpretation is not satisfactory to claim that the finite N YM$_2$ is equivalent to a string theory. In this section, we will refine the discussion in Ref. [13] and give a string theoretical interpretation of the partition function for the finite N case. We will also mention the connection between the finite N theory and the large N theory.

Let us first recall the geometrical interpretation for the finite N YM$_2$ given in Ref. [13]. Using the same technique as that for the large N theory, we can rewrite the partition function of the finite N theory (2.1) as

$$
Z_{\mathrm{YM}}(G, \lambda A, N) = \sum_{n=1}^{\infty} e^{-\frac{n\lambda A}{2}} \sum_{i,t,h=0}^{\infty} \frac{(-1)^i (\lambda A)^{i+t+h}}{i!\, t!\, h!} \left(\frac{n(n-1)}{2}\right)^t \left(\frac{n}{2}\right)^h N^{n(2-2G)-i-2t-2h}
$$

$$
\times \sum_{s_1,t_1,\cdots,s_G,t_G \in S_n} \sum_{p_1,\cdots,p_i \in T_2} \sum_{R \in Y_n^N} \frac{d_R}{(n!)^2} \chi_R(p_1\cdots p_i \Omega_n^{2-2G} \prod_{j=1}^{G} s_j t_j s_j^{-1} t_j^{-1}), \quad (3.1)
$$

where Y_n^N is the set of Young diagrams with n boxes and less than N rows. Note that Y_n^N is equal to Y_n if n is less than N. To interpret (3.1) geometrically, we must express the sum of representations in a form of the delta function of products of elements of S_n. To this end, the authors of Ref. [13] defined the projection operator:[3]

$$
P_n^{(N)} \equiv \sum_{\rho \in S_n} \sum_{R \in Y_n^N} \frac{d_R}{n!} \chi_R(\rho)\rho \equiv \sum_{\rho \in S_n} N^{K_\rho - n} P^{(N)}(\rho)\rho, \quad (3.2)
$$

which has the property

$$
\delta(\sigma P_n^{(N)}) = \sum_{R \in Y_n^N} \frac{d_R}{n!} \chi_R(\sigma), \quad (3.3)
$$

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[3]Use of the formula

$$
\sum_{\sigma \in S_n} \chi_R(\sigma)\chi_R(\sigma^{-1}\tau) = \frac{n!}{d_R} \chi_R(\tau),
$$

makes it straightforward to show that $P_n^{(N)}$ is indeed a projection operator, that is, $\left(P_n^{(N)}\right)^2 = P_n^{(N)}$.

for any $\sigma \in S_n$. Then (3.1) can be rewritten as

$$Z_{\mathrm{YM}}(G, \lambda A, N) = \sum_{n=1}^{\infty} e^{-\frac{n\lambda A}{2}} \sum_{i,t,h=0}^{\infty} \frac{(-1)^i (\lambda A)^{i+t+h}}{i!\, t!\, h!} \left(\frac{n(n-1)}{2}\right)^t \left(\frac{n}{2}\right)^h N^{n(2-2G)-i-2t-2h}$$

$$\times \sum_{s_1,t_1,\cdots,s_G,t_G \in S_n} \sum_{p_1,\cdots,p_i \in T_2} \frac{1}{n!} \delta(p_1 \cdots p_i \Omega_n^{2-2G} P_n^{(N)} \prod_{j=1}^{G} s_j t_j s_j^{-1} t_j^{-1}). \qquad (3.4)$$

The only difference between this expression and the chiral sector of the large N theory (2.9) is the existence of the projection operator $P_n^{(N)}$ in the delta function. In Ref. [13], the projection operator is interpreted as a sum of multiple branch points (the projection point) on the covering sheets, and thus, an element ρ in $P_n^{(N)}$ contributes $(K_\rho - n)$ to the Euler characteristic.[4] Therefore, it is possible to give a geometrical interpretation for the partition function (3.4).

However, there are some difficulties in claiming that finite N YM_2 is equivalent to a closed string theory through the above interpretation: first, the partition function (3.1) contains the terms with odd powers of $1/N$; second, we do not have any candidate for a string interaction which provides a contribution N^{n-K_ρ} to the partition function; and third, the relationship to the large N theory is not clear. We here give a way to overcome these difficulties by separating the partition function of the finite N YM_2 into two sectors, which we call "perturbative sector" and "residual sector", as

$$Z_{\mathrm{YM}}(G, \lambda A, N) = Z_{\mathrm{YM}}^{\mathrm{pert}}(G, \lambda A, N) + Z_{\mathrm{YM}}^{\mathrm{res}}(G, \lambda A, N), \qquad (3.5)$$

where

$$Z_{\mathrm{YM}}^{\mathrm{pert}}(G, \lambda A, N) \equiv \sum_{n=1}^{N-1} \sum_{R \in Y_n} (\dim R)^{2-2G} \exp\left[-\frac{\lambda A}{2N} C_2(R)\right], \qquad (3.6)$$

$$Z_{\mathrm{YM}}^{\mathrm{res}}(G, \lambda A, N) \equiv \sum_{n=N}^{\infty} \sum_{R \in Y_n^N} (\dim R)^{2-2G} \exp\left[-\frac{\lambda A}{2N} C_2(R)\right]. \qquad (3.7)$$

For the purposes of subsequent discussion, we will translate (3.6) and (3.7) into the language

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[4]In Ref. [13], it is discussed that the N^{n-K_ρ} which compensates the N dependence comes from an unknown string interaction.

of the symmetry group for $G = 1$:

$$Z_{\text{YM}}^{\text{pert}}(G = 1, \lambda A, N) = \sum_{n=1}^{N-1} e^{-\frac{n\lambda A}{2}} \sum_{c,t,h=0}^{\infty} \frac{(\lambda A)^{2c+t+h}}{(2c)!\, t!\, h!} \left(\frac{n(n-1)}{2}\right)^t \left(\frac{n}{2}\right)^h N^{-2(c+t+h)}$$

$$\times \sum_{s,t\in S_n} \sum_{p_1,\cdots,p_{2c}\in T_2} \frac{1}{n!} \delta(p_1 \cdots p_{2c} s t s^{-1} t^{-1}) \tag{3.8}$$

$$Z_{\text{YM}}^{\text{res}}(G = 1, \lambda A, N) = \sum_{n=N}^{\infty} e^{-\frac{n\lambda A}{2}} \sum_{i,t,h=0}^{\infty} \frac{(-1)^i (\lambda A)^{i+t+h}}{i!\, t!\, h!} \left(\frac{n(n-1)}{2}\right)^t \left(\frac{n}{2}\right)^h N^{-i-2t-2h}$$

$$\times \sum_{s,t\in S_n} \sum_{p_1,\cdots,p_i\in T_2} \sum_{R\in Y_n^N} \frac{d_R}{(n!)^2} \chi_R(p_1 \cdots p_i s t s^{-1} t^{-1}) . \tag{3.9}$$

To derive (3.8), we have used the fact that there exists the transpose representation R^T for any representation R in (3.6), as in the case of the chiral sector (2.6).

The partition function (3.6) contains only even powers of $1/N$, thus we can interpret this as a sum over covering maps from a set of closed string world-sheets by repeating the consideration described in the previous section. This sector takes the same form as that of the chiral sector of the large N theory, except that the range for the summation is now restricted to $N-1$. This means that (3.6) corresponds to the perturbative expansion of a closed string theory, which is the reason why we refer to (3.6) as the "perturbative" sector. The restriction of the range for the summation tells us that there is an upper bound to the wrapping number of the world-sheets, which is a manifestation of the stringy exclusion principle [24]. Moreover, the coefficients of the $1/N$ expansion defined by (3.6) become those of the chiral sector (2.9) in the large N limit. From these facts, we claim that the perturbative sector of the partition function in the finite N case corresponds to the chiral sector of the full large N theory.

On the other hand, we cannot give a satisfactory closed string interpretation of the residual sector since odd powers of $1/N$ exist in this sector. However, it is natural to consider that the residual sector should contain corrections to the perturbative expansion described by the perturbative sector. In fact, there are some signs in the explicit expression (3.9) which support this expectation. The most suggestive is the overall factor $e^{-\frac{N\lambda A}{2}}$ in the free energy[5]. This is compatible with the fact that non-perturbative effects of a string theory appear to the string partition function as $e^{-\mathcal{O}(1/g_s)}$. Moreover, recent developments in the non-perturbative aspects of string theories have elucidated that the non-perturbative effects come from D-branes. Since there are open strings on the D-branes, there should be odd powers of g_s in the partition function of string theory if we take into account the non-perturbative effects. From this observation, the presence of the odd powers of $1/N$ in (3.7) would rather suggest that we should consider open string world-sheets to give a string theoretical interpretation of

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[5]It is easy to see that the overall factor $e^{-N\lambda A/2}$ is ubiquitous in the connected part of the residual sector partition function.

the residual sector. The above considerations lead us to conjecture that the residual sector (3.9) contains contributions from open string world-sheets.[6] To prove this, however, we must show that the residual sector indeed contains information which can be interpreted as a map from open string world-sheets. This is important future work [25].

We conclude this section by considering another way to divide the partition function (3.1). If one respects the even powers of $1/N$ in the partition function, one may restrict the sum of representations to those whose corresponding Young diagrams can be put in the $N \times N$ squared box. Let us express the set of such diagrams with n boxes as $Y_n^{N \times N}$. Since any Young diagram R in $Y_n^{N \times N}$ has the transpose diagram R^T in $Y_n^{N \times N}$, the restricted partition function becomes ($G = 1$, for simplicity)

$$
\begin{aligned}
Z_{\mathrm{YM}}^{N \times N}(G = 1, \lambda A, N) &= \sum_{n=1}^{N^2} \sum_{R \in Y_n^{N \times N}} \exp\left[-\frac{\lambda A}{2N} C_2(R) \right] \\
&= \sum_{n=1}^{N^2} e^{-\frac{n \lambda A}{2}} \sum_{c,t,h=0}^{\infty} \frac{(\lambda A)^{2c+t+h}}{(2c)!\, t!\, h!} \left(\frac{n(n-1)}{2} \right)^t \left(\frac{n}{2} \right)^h N^{-2(c+t+h)} \\
&\quad \times \sum_{s,t \in S_n} \sum_{p_1,\cdots,p_{2c} \in T_2} \sum_{R \in Y_n^{N \times N}} \frac{d_R}{(n!)^2} \chi_R(p_1 \cdots p_{2c} s t s^{-1} t^{-1}) .
\end{aligned}
\tag{3.10}
$$

However, to provide a geometrical interpretation, we must introduce a projection operator,

$$
P_n^{N \times N} \equiv \sum_{\rho \in S_n} \sum_{R \in Y_n^{N \times N}} \frac{d_R}{n!} \chi_R(\rho) \rho,
\tag{3.11}
$$

and rewrite (3.10) as

$$
\begin{aligned}
Z_{\mathrm{YM}}^{N \times N}(G = 1, \lambda A, N) &= \sum_{n=1}^{N^2} e^{-\frac{n \lambda A}{2}} \sum_{c,t,h=0}^{\infty} \frac{(\lambda A)^{2c+t+h}}{(2c)!\, t!\, h!} \left(\frac{n(n-1)}{2} \right)^t \left(\frac{n}{2} \right)^h N^{-2(c+t+h)} \\
&\quad \times \sum_{s,t \in S_n} \sum_{p_1,\cdots,p_{2c} \in T_2} \frac{1}{n!} \delta(P_n^{N \times N} p_1 \cdots p_{2c} s t s^{-1} t^{-1}) .
\end{aligned}
\tag{3.12}
$$

Since, we interpret an element of S_n in the delta function as an existence of a multiple branch point on the covers, it is natural to assume that the factor N^{-n+K_ρ} accompanies the element $\rho \in S_n$ in the projection operator (3.11) like as in the case of the finite N partition function (3.4). Thus, terms with odd powers of $1/N$ still remain in the restricted partition function (3.10), which is hard to be interpreted as a perturbative closed string theory. This analysis indicates that the way of dividing the partition function that we have adopted in this paper would be a more proper way to separate the perturbative and non-perturbative effects of the dual string theory.

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[6]For another discussion on the non-perturbative string effects in YM$_2$, see Ref. [23].

4   Conclusion and discussion

In this paper, we discussed the equivalence of the finite N YM$_2$ to a string theory. YM$_2$ with the gauge group $SU(N)$ is exactly solvable and the partition function is expressed as a sum of a function of the irreducible representation R of $SU(N)$ (see (2.1)). We separated the partition function into two sectors by introducing a cutoff parameter N to the number of boxes of the corresponding Young diagram. We demonstrated that the sector that is defined by the sum of diagrams with boxes less than N (the perturbative sector) corresponds to the perturbative expansion of a closed string theory. The wrapping number of the closed strings that constitute this sector is restricted to less than N, which is a manifestation of the stringy exclusion principle. We also showed that this sector becomes the chiral sector [11] in the large N limit. We presented the conjecture that we can also give a string theoretical interpretation of the other sector (the residual sector) by taking into account non-perturbative effects of the dual string theory. We proposed some evidence in support of this conjecture.

Finally, we make some comments. Although we defined the finite N counterpart of one of the chiral sectors of the large N theory, we do not know yet how the other chiral sector appears as a large N limit of the finite N YM$_2$. Recalling the way of construction of the composite diagram, we see that one of the chiral sectors comes from a sum over the set of Young diagrams $\{R\}$ with a finite number of boxes and the other chiral sector comes from a sum over the set of the conjugate diagrams $\{\bar{S}\}$ of Young diagrams $\{S\}$ with a finite number of boxes. In the large N theory, we must take these summation independently since $\bar{S}$ has an infinite number of boxes. For the finite N theory, however, the number of boxes in $\bar{S}$ is also finite, and thus, the summation in (2.1) would contain both of the chiral sectors. From this consideration, it is plausible to assume that the residual sector (3.7) also contains the finite N counterpart of the other chiral sector, not only non-perturbative corrections to the perturbative expansion. In order to show this, we must separate contributions from perturbative and non-perturbative effects of the dual string theory in the residual sector. As one possibility, non-perturbative effects could be evaluated as a sum over open string world-sheets whose boundaries shrink to points. This is expected because D-branes would excite on the target space if the string coupling constant is finite and they make holes to the world-sheets. Note that the boundaries must shrink to points because of the area preserving diffeomorphism invariance of YM$_2$. Another advantage of assuming the D-branes is that they might suppress the wrapping number of world-sheets because the appearance of D-branes would decrease the free energy of the system when the wrapping number of the world-sheets becomes too large. This might provide a reason for the string exclusion principle being realized in the dual string theory of the finite N YM$_2$. We hope to confirm this scenario and to report subsequent developments in the near future [25].

Acknowledgement

A   Geometric interpretation of the YM_2 partition function

In this appendix, we review the geometric interpretation for the partition function of the chiral sector. For a complete discussion, see Refs. [11, 12].

The starting point is the partition function (2.9). In the following, show that (2.9) is equal to a sum over covering maps from a set of (disconnected) two-dimensional manifolds to the target space $\mathcal{M}$ [10, 11, 12]. As a preparation, let us consider a n-fold covering map ν with i single branch points at $\{w_1, \cdots, w_i\} \in \mathcal{M}$ and $|2-2G|$ multiple branch points (twists points) at $\{z_1, \cdots, z_{|2-2G|}\} \in \mathcal{M}$. We assume that ν also maps some handles of the sheets to points on $\mathcal{M}$.[7] Since there are n sheets over a point on the target space, we classify such a mapped handle into two types; one connects two different sheets (a tube), and the other is on a same sheet (a contracted handle) [11]. Here, we fix the number of tubes and contracted handles to t and h, respectively. In addition, we assume that the positions of the branch points, the tubes and the contracted handles can move over $\mathcal{M}$, while those of the multiple branch points are fixed.

Under the above setting, we evaluate (1) the sum of the Euler characteristic of the sheets on $\mathcal{M}$, (2) the numerical factor that comes from integrating over the parameters of the map ν, and (3) the symmetry factor of the map.

(1) Euler characteristic

If there are no branch points, tubes and contracted handles, the Euler characteristic of the sheet is $n(2-2G)$ since the map ν covers the target space n times. Thus, the sum of the Euler characteristic of the set of the sheets on $\mathcal{M}$ is

$$2 - 2g = n(2-2G) - i - 2(t+h) - \sum_{k=1}^{|2-2G|} \omega_k, \tag{A.13}$$

since each of a branch point, a tube and a contracted handle contributes -1, -2 and -2 to the Euler characteristic, respectively. Here, ω_k is the contribution to the Euler characteristic from the k'th multiple branch point.

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[7]This map must not break the area preserving diffeomorphism of YM_2.

(2) Numerical factor

From the assumption that the position of the single branch points, the tubes and contracted handles can move on the target space, the factor $(\lambda A)^{i+t+h}/i!\,t!\,h!$ appears by integrating over their positions.[8] In addition, a tube and a contracted handle can attach to the sheets in $n(n-1)/2$ and $n/2$ possible ways, respectively. As a result, the numerical factor accompanied by ν is

$$\frac{(\lambda A)^{i+t+h}}{i!\,t!\,h!}\left(\frac{n(n-1)}{2}\right)^t\left(\frac{n}{2}\right)^h. \tag{A.14}$$

(3) Symmetry factor

Let us consider the fundamental group on $\mathcal{G}\equiv\mathcal{M}\backslash\{w_1,\cdots,w_i,z_1,\cdots,z_{|2-2G|}\}$. This group is generated by a set of loops $\{a_1,b_1,\cdots,a_G,b_G\}$ around the alpha and beta cycles of $\mathcal{M}$, a set of loops $\{c_1,\cdots,c_i\}$ around the branch points, and a set of loops $\{d_1,\cdots,d_{|2-2G|}\}$ around the twists points, which are characterized by the relationship,

$$c_1\cdots c_i d_1\cdots d_{|2-2G|}\prod_{j=1}^{G}a_j b_j a_j^{-1}b_j^{-1}=1. \tag{A.15}$$

We can define a homomorphism from the fundamental group on $\mathcal{G}$ to the symmetry group S_n by mapping a loop to a permutation of n-sheets. Thus, we see that the expression,

$$\sum_{s_1,t_1,\cdots,s_G,t_G\in S_n}\;\sum_{p_1,\cdots,p_i\in T_2}\frac{1}{n!}\delta(p_1\cdots p_i\Omega_n^{2-2G}\prod_{j=1}^{G}s_j t_j s_j^{-1}t_j^{-1}) \tag{A.16}$$

is equal to $\sum_{\nu\in\Sigma(G,n,i)}1/|S_\nu|$, where $|S_\nu|$ is the symmetry factor of the n-fold cover ν. Note that the factor $n!$ in (A.16) is the number of possible labelings of n sheets.

Combining the above observations (1), (2) and (3), we see that the partition function (2.6) is regarded as a sum over the covering maps with the properties described at the beginning of this section. This provides strong evidence for the equivalence of the large N YM$_2$ and a two-dimensional string theory.

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